THE CREMMER-GERVAIS SOLUTION OF THE YANG BAXTER EQUATION

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Abstract. A direct proof is given of the fact that the Cremmer-Gervais R-matrix satisfies the (Quantum) Yang-Baxter equation

1. Introduction

Let V be a vector space of rank n over a field F. Let $c \in \operatorname{End} V \otimes V$ be a linear operator. Define $c_{12}, c_{23} \in \operatorname{End} V \otimes V \otimes V$ by $c_{12} = c \otimes Id$, $c_{23} = Id \otimes c$. Then c is said to satisfy the Yang-Baxter equation (YBE) if

$$c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23}$$

An extremely interesting solution of this equation was found by Cremmer and Gervais in their paper [1]. In its slightly more general two parameter form, it is (up to a scalar)

$$c(e_i \otimes e_j) = \begin{cases} qe_j \otimes e_i & \text{if } i = j \\ qp^{i-j}e_j \otimes e_i + \sum_{i \le k < j}(q - q^{-1})p^{i-k}e_k \otimes e_{i+j-k} & \text{if } i < j \\ q^{-1}p^{i-j}e_j \otimes e_i + \sum_{j < k < i}(q^{-1} - q)p^{i-k}e_k \otimes e_{i+j-k} & \text{if } i > j \end{cases}$$

where $\{e_1, \dots, e_n\}$ is a basis for V and q and p are non-zero elements of F. Taking $q = p^{n/2}$ yields the original operator given by Cremmer and Gervais. The derivation of this solution used some fairly technical calculations involving chiral vertex operators and is a litte inaccessible to the non-specialist. Here we give an elementary proof of this result along the same lines as the proof in [3] for the standard solutions of the Yang-Baxter equation.

2. Linear combinations of solutions of the YBE

Suppose f and g are solutions of the YBE and let $\alpha, \beta \in F$. Expanding out the equation

$$(\alpha f + \beta g)_{12}(\alpha f + \beta g)_{23}(\alpha f + \beta g)_{12} = (\alpha f + \beta g)_{23}(\alpha f + \beta g)_{12}(\alpha f + \beta g)_{23}$$

we see that $c = \alpha f + \beta g$ will be a solution of the YBE for all $\alpha, \beta \in A$ if the following two conditions are satisfied:

$$f_{12}g_{23}g_{12} + g_{12}f_{23}g_{12} + g_{12}g_{23}f_{12} = f_{23}g_{12}g_{23} + g_{23}f_{12}g_{23} + g_{23}g_{12}f_{23}$$

$$g_{12}f_{23}f_{12} + f_{12}g_{23}f_{12} + f_{12}f_{23}g_{12} = g_{23}f_{12}f_{23} + f_{23}g_{12}f_{23} + f_{23}f_{12}g_{23}$$

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In the case where f is the permutation operator $P(e_i \otimes e_j) = e_j \otimes e_i$, the second condition is true for any g (since $g_{12}P_{23}P_{12} = P_{23}P_{12}g_{23}$ and similar equalities hold for the other terms). Thus we obtain the following simple condition which we shall refer to as the *compatibility condition.*

Lemma 2.1. *Suppose that $g \in \operatorname{End} V \otimes V$ is a solution of the YBE. Then $c = \alpha P + \beta g$ will be a solution of the YBE for all $\alpha, \beta \in F$ if*

$$g_{12}g_{23}P_{12} + g_{12}P_{23}g_{12} + P_{12}g_{23}g_{12} = g_{23}g_{12}P_{23} + g_{23}P_{12}g_{23} + P_{23}g_{12}g_{23}. \tag{2.1}$$

We shall apply this result to the case where

$$g(e_i \otimes e_j) = \sum_k \eta(i,j,k) e_k \otimes e_{i+j-k} \tag{2.2}$$

and

$$\eta(i,j,k) = \begin{cases} 1 & \text{if } i \leq k < j \\ -1 & \text{if } j \leq k < i \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Taking $\alpha = q$ and $\beta = (q - q^{-1})$ yields

$$c(e_i \otimes e_j) = \begin{cases} q e_j \otimes e_i & \text{if } i = j \\ q e_j \otimes e_i + \sum_{i \leq k < j} (q - q^{-1}) e_k \otimes e_{i+j-k} & \text{if } i < j \\ q^{-1} e_j \otimes e_i + \sum_{j < k < i} (q^{-1} - q) e_k \otimes e_{i+j-k} & \text{if } i > j \end{cases}$$

which is the Cremmer-Gervais operator described in the introduction in the case where $p = 1$. Once we have shown that this operator satisfies the Yang-Baxter equation, it follows from some well-known "twisting" results [2] that the more general operator is also a solution.

3. The compatibility condition

In this section we check the compatibility condition (2.1) for the operator g given above.

Lemma 3.1. *Let $g \in \operatorname{End} V \otimes V$ be an operator of the form*

$$g(e_i \otimes e_j) = \sum_k \eta(i,j,k) e_k \otimes e_{i+j-k}$$

where $\eta(i,j,k) = 0$ if k is not between i and j. Then the condition of Lemma 2.1 is satisfied if and only if

$$\begin{aligned} \eta(i,k,a+b-j)\eta(j,a+b-j,a) + \eta(i,j,b+a-k)\eta(b+a-k,k,a) \\ + \eta(i,j,b)\eta(i+j-b,k,a) = \eta(i,k,a)\eta(i+k-a,j,b) \\ + \eta(j,k,a)\eta(i,j+k-a,b) + \eta(j,k,j+k-b)\eta(i,j+k-b,a) \end{aligned} \tag{3.1}$$

for all $i,j,k,a,b \in \{1,2,\ldots,n\}$.

Proof. Let $d_l = g_{12}g_{23}P_{12} + g_{12}P_{23}g_{12} + P_{12}g_{23}g_{12}$ and $d_r = g_{23}g_{12}P_{23} + g_{23}P_{12}g_{23} + P_{23}g_{12}g_{23}$. Denote $e_i \otimes e_j \otimes e_k$ by $[ijk]$. Then

$$\begin{aligned} d_l[ijk] = &\sum_{s,t} \eta(i,k,t)\eta(j,t,s)[s, j+t-s, i+k-t] \\ &+ \sum_{s,t} \eta(i,j,s)\eta(s,k,t)[t, s+k-t, i+j-s] \\ &+ \sum_{s,t} \eta(i,j,s)\eta(i+j-s,k,t)[t,s,i+j+k-s-t] \end{aligned}$$

and

$$\begin{aligned} d_r[ijk] = &\sum_{s,t} \eta(i,k,s)\eta((k+i-s,j,t)[s,t,i+j+k-s-t] \\ &+ \sum_{s,t} \eta(j,k,s)\eta(i,j+k-s,t)[s,t,i+j+k-t-s] \\ &+ \sum_{s,t} \eta(j,k,s)\eta(i,t,s)[s,j+k-t,i+t-s] \end{aligned}$$

Comparing the coefficients of $[a,b,i+j+k-a-b]$ then yields the result. $\square$

Now set

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

and

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}.$$

Notice that $\eta(i,j,k) = u(k-i) - u(k-j)$.

Lemma 3.2. *For any integers a, b, i, j, k,*

$$\begin{aligned} &u(a+b-i-j)(u(a-j)+u(b-i)-u(b-j)-u(j-b))+u(k-b)u(a+b-i-k)) \\ &= u(a-i)(u(k-b)-u(j-b)-u(b-j)+u(b+a-i-k))+u(b-i)u(a-j) \end{aligned}$$

Proof. First note that

$$u(x) + u(-x) = 1 + \delta(x)$$

and

$$u(x+y)(u(x)+u(y)) = u(x)u(y) + u(x+y).$$

From this it follows that

$$u(a+b-i-j)((u(a-j)+u(b-i)) = u(a-j)u(b-i)+u(a+b-i-j)$$
$$u(a+b-i-j)(-u(b-j)-u(j-b)) = -u(a+b-i-j) - \delta(b-j)u(a-i)$$

and

$$\begin{aligned} u(k-b)u(a+b-i-k) &= (1-u(b-k)+\delta(b-k))u(a+b-i-k) \\ &= u(a+b-i-k)(1-u(b-k)) + \delta(b-k)u(a-i) \\ &= u(a+b-i-k)u(a-i) - u(a-i)u(b-k) + \delta(b-k)u(a-i) \\ &= u(a+b-i-k)u(a-i) + u(a-i)(u(k-b)-1). \end{aligned}$$

Combining these equation yields the desired result. $\square$

Theorem 3.3. *For η as defined in (2.3), the operator $g(e_i \otimes e_j) = \sum_k \eta(i,j,k) e_k \otimes e_{i+j-k}$ satisifies the compatibility condition (2.1).*

Proof. Expanding out the left hand side of (3.1) using $\eta(i,j,k) = u(k-i) - u(k-j)$ yields

$$\begin{aligned} &u(a+b-j-k)[u(a-k) - u(k-b) + u(j-b) - u(a-j)] \\ &+ u(a+b-i-j)[-u(j-b) + u(a-j) + u(b-i) - u(b-j)] \\ &+ u(a+b-i-k)[u(k-b) - u(a-k)] \\ &+ u(a-k)[u(b-j) - u(b-i)]. \end{aligned}$$

Similarly the right hand side becomes

$$\begin{aligned} &u(a+b-j-k)[u(a-k) - u(k-b) + u(j-b) - u(a-j)] \\ &+ u(a-k)[u(b-j) - u(b-i) - u(a+b-i-k)] \\ &+ u(a-i)[u(k-b) - u(b-j) - u(j-b) + u(a+b-i-k)] \\ &+ u(b-i)u(a-j). \end{aligned}$$

The equality of these two expressions follows from the identity in Lemma 3.2. □

4. The Yang-Baxter equation

In this section we verify that the operator $g(e_i \otimes e_j) = \sum_k \eta(i,j,k) e_k \otimes e_{i+j-k}$ given above satisfies the Yang-Baxter equation. We begin by converting the problem into an identity for η.

Lemma 4.1. *Let $g \in \operatorname{End} V \otimes V$ be an operator of the form*

$$g(e_i \otimes e_j) = \sum_k \eta(i,j,k) e_k \otimes e_{i+j-k}$$

where $\eta(i,j,k) = 0$ if k is not between i and j. Then g saisfies the Yang-Baxter equation if and only if

$$\sum_a \eta(j,k,a)\eta(i,a,c)\eta(i+a-c, j+k-a, h) = \sum_s \eta(i,j,s)\eta(i+j-s, k, h+c-s)\eta(s, h+c-s, c) \tag{4.1}$$

for all $i,j,k,c,h \in \{1,2,\ldots,n\}$.

Proof. The left hand side of (4.1) is the coefficient of $e_c \otimes e_h \otimes e_{i+j+k-c-h}$ in the expansion of $g_{23}g_{12}g_{23}(e_i \otimes e_j \otimes e_k)$. Similarly the right hand side is the coefficient of $e_c \otimes e_h \otimes e_{i+j+k-c-h}$ in $g_{12}g_{23}g_{12}(e_i \otimes e_j \otimes e_k)$. □

The following identities are used in the proof of the next three results.

Lemma 4.2. *For integers a,b,c,d,e,*

1. $\eta(a+d, b+d, c+d) = \eta(a,b,c)$.
2. $\eta(a,b,c) = -\eta(b,a,c)$
3. $\eta(a,b,c) = \eta(-b,-a,-c-1) = \eta(a,b,a+b-c-1)$
4. $\eta(a,a+1,c) = \delta(a-c)$
5. $\sum_a \eta(b,c,a) = c-b$
6. $\eta(a,b,d) + \eta(b,c,d) = \eta(a,c,d)$
7. $\eta(a,b+1,c)\eta(c,a,b) = 0$

8. $\eta(a,b,c)\eta(c,b,d) = \eta(a,b,d)\eta(a,d+1,c)$
9. $\eta(a,b,c)\eta(d,c,e) = \eta(a,b,c)\eta(d,a,e) + \eta(a,b,e)\eta(e+1,b,c)$

Proof. The proofs are either trivial or routine calculations. □

Lemma 4.3. *For any integers, t, s, b, d, h,*

$$\sum_a \eta(t,s,a)\eta(b+a,d-a,h) = (s-t)\eta(b+t,d-t,h) \\ + (d-h-s)\eta(d-s,d-t,h) + (h-b-s+1)\eta(b+t,b+s,h)$$

Proof. Using the identities of Lemma 4.2,

$$\begin{aligned}
&\sum_a \eta(t,s,a)\eta(b+a,d-a,h) \\
&= -\sum_a \eta(t,s,a)\eta(d-b-a,a,h-b) \\
&= -\sum(\eta(t,s,a)\eta(d-b-a,t,h-b) + \eta(t,s,h-b)\eta(h-b+1,s,a)) \\
&= (h-b-s+1)\eta(b+t,b+s,h) - \sum_a \eta(t,s,a)\eta(d-b-t,a,d-h-1)
\end{aligned}$$

Now

$$\begin{aligned}
&-\sum_a \eta(t,s,a)\eta(d-b-t,a,d-h-1) \\
&= -\sum_a \eta(t,s,a)\eta(d-b-t,t,d-h-1) + \eta(t,s,d-h-1)\eta(d-h,s,a) \\
&= (s-t)\eta(t+b-d,-t,h-d) + (d-s-h)\eta(-s,-t,h-d) \\
&= (s-t)\eta(b+t,d-t,h) + (d-h-s)\eta(d-s,d-t,h)
\end{aligned}$$

Combining these two equations yields the assertion. □

Lemma 4.4. *For any integers i, j, k, c, h,*

$$\begin{aligned}
&\sum_a \eta(j,k,a)\eta(i,a,c)\eta(i+a-c,j+k-a,h) \\
&= \eta(j,k,c)((k-c-1)\eta(i-c+k,j+k-c,h) + (j-h)\eta(j,j+k-c,h) \\
&\qquad + (h-i)\eta(i,i+k-c,h)) \\
&+ \eta(i,j,c)((c-i+1)\eta(i+j-c,i+k-c,h) + (h-j)\eta(i+j-c,j,h) \\
&\qquad + (k-h)\eta(i+k-c,k,h))
\end{aligned}$$

Proof. By part 7 of Lemma 4.2

$$\begin{aligned}
&\sum_a \eta(j,k,a)\eta(i,a,c)\eta(i+a-c,j+k-a,h) \\
&= \sum_a(\eta(j,k,a)\eta(i,j,c) + \eta(j,k,c)\eta(c+1,k,a))\eta(i+a-c,j+k-a,h)
\end{aligned}$$

Using Lemma 4.3 we obtain that

$$\begin{aligned}&\sum_a \eta(j,k,a)\eta(i,j,c)\eta(i+a-c,j+k-a,h)\\&=\eta(i,j,c)((k-j)\eta(i+j-c,k,h)+(j-h)\eta(j,k,h)\\&\quad+(h+c-i-k+1)\eta(i+j-c,i+k-c,h))\end{aligned}$$

and that

$$\begin{aligned}&\sum_a \eta(j,k,c)\eta(c+1,k,a)\eta(i+a-c,j+k-a,h)\\&=\eta(j,k,c)((k-c-1)\eta(i+1,j+k-c-1,h)+(j-h)\eta(j,j+k-c-1,h)\\&\quad+(h+c-i-k+1)\eta(i+1,i+k-c,h))\\&=\eta(j,k,c)((k-c-1)\eta(i,j+k-c,h)+(j-h)\eta(j,j+k-c,h)\\&\quad+(h+c-i-k+1)\eta(i,i+k-c,h))\end{aligned}$$

Using these formulas and repeated application of the identity

$$\eta(a,b,h)+\eta(b,c,h)=\eta(a,c,h)$$

yields the result. □

Theorem 4.5. *For η as defined in (2.3), the operator $g(e_i\otimes e_j)=\sum_k \eta(i,j,k)e_k\otimes e_{i+j-k}$ satisifies the Yang-Baxter equation.*

Proof. Let

$$\zeta(i,j,k,c,h)=\sum_a \eta(j,k,a)\eta(i,a,c)\eta(i+a-c,j+k-a,h)$$

(the left hand side of equation (4.1)). It is easily verified that the right hand side of equation (4.1) is then $\zeta(i+j-k,i,j,h+c-k,i+j-h)$. Now

$$\begin{aligned}&\zeta(i+j-k,i,j,h+c-k,i+j-h)\\&=(j+k-h-c-1)\eta(j,k,c)\eta(i-c+k,j+k-c,h)\\&\quad+(h-j)\eta(h,j+k-c,k)\eta(i-c+k,j+k-c,h)\\&\quad+(k-h)\eta(h,j+k-c,k)\eta(i-c+k,j+k-c,h)\\&\quad+(h+c-i-j+1)\eta(i,j,c)\eta(i+j-c,i+k-c,h)\\&\quad+(j-h)\eta(i+j-c,h,j)\eta(i+j-c,i+k-c,h)\\&\quad+(h-i)\eta(i+j-c,h,i)\eta(i+j-c,i+k-c,h)\end{aligned}$$

We may then rearrange these terms one at a time using Proposition 4.2:

$$\begin{aligned}&(j+k-h-c-1)\eta(j,k,c)\eta(i-c+k,j+k-c,h)\\&=(k-c-1)\eta(j,k,c)\eta(i-c+k,j+k-c,h)\\&\quad+(j-h)(j+k-h-c-1)\eta(j,k,c)\eta(i-c+k,j+k-c,h)\end{aligned}$$

and

$$\begin{aligned}&(h-j)\eta(h,j+k-c,k)\eta(i-c+k,j+k-c,h)\\&=(h-j)\eta(i+k-c,j+k-c,j)\eta(i+k-c,j+1,h)\\&=(h-j)\eta(i+k-j,k,c)\eta(i+k-c,j,h).\end{aligned}$$

Similarly

$$(k-h)\eta(h,j+k-c,k)\eta(i-c+k,j+k-c,h) = (k-h)\eta(i,j,c)\eta(i+k-c,k,h),$$

$$\begin{aligned}(h+c-i-j+1)&\eta(i,j,c)\eta(i+j-c,i+k-c,h)\\ &= (c-i+1)\eta(i,j,c)\eta(i+j-c,i+k-c,h)\\ &\quad + (h-j)\eta(i,j,c)\eta(i+j-c,i+k-c,h),\end{aligned}$$

$$(j-h)\eta(i+j-c,h,j)\eta(i+j-c,i+k-c,h) = (j-h)\eta(i+k-j,i,c)\eta(i+k-c,j,h),$$

and

$$(h-i)\eta(i+j-c,h,i)\eta(i+j-c,i+k-c,h) = (h-i)\eta(k,j,c)\eta(i+k-c,i,h)$$

Adding these terms and rearranging easily yields $\zeta(i,j,k,c,h)$ as required. □

Finally we make some observations about invertibility and the Hecke condition. Recall that R is said to be *Hecke* if it satisfies the condition

$$(R-q)(R+q^{-1}) = 0$$

for some q.

Lemma 4.6. 1. $g^2 = g$
2. $gP = -g$
3. $Pg = g + P - I$

Proof. The first part follows from the identity

$$\sum_k \eta(i,j,k)\eta(k,i+j-k,l) = \eta(i,j,l)$$

which is a consequence of Lemma 4.3. The second and third parts follow from the identities $\eta(j,i,k) = -\eta(i,j,k)$ and $\eta(i,j,i+j-k) = \eta(i,j,k) + \delta(k-j) - \delta(k-i)$ respectively. □

Proposition 4.7. *Let α and β be non-zero elements of F. The operator $R = \alpha P + \beta g$ is invertible if and only if $\alpha \neq \beta$. It is Hecke if and only if $\beta = \alpha - \alpha^{-1}$.*

Proof. Using Lemma 4.6 we find that

$$R^2 = \beta R + \alpha(\alpha-\beta)I$$

and the proposition then follows immediately. □

Theorem 4.8. *Let F be a field and let V be a vector space with basis $\{e_1,\dots,e_n\}$. Let $c \in \operatorname{End} V \otimes V$ be the linear operator*

$$c(e_i \otimes e_j) = qp^{i-j} e_j \otimes e_i + \sum_k (q-q^{-1})p^{i-k}\eta(i,j,k) e_k \otimes e_{i+j-k}$$

Then c is an invertible solution of the Yang-Baxter equation.

Proof. For $p = 1$ the result follows from Lemma 2.1, Theorem 3.3, Proposition 4.7 and Theorem 4.5. For more general p we apply [2, Theorem 3.3]. □

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